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Connes' spectral triple and $U(1)$ gauge theory on finite sets

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Abstract

We first apply Connes' noncommutative geometry to a finite point set. The explicit form of the action functional of $U(1)$ gauge field on this n-point set is obtained. We then construct the $U(1)$ gauge theory on a disconnected manifold consisting of n copies of a given manifold. In this case, the explicit action functional of $U(1)$ gauge field is also obtained. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Within the framework of Connes' noncommutative geometry [1,2] (for the brief introduction, see also [3–5]), the Higgs field and the symmetry breaking mechanism in the standard model have a remarkable geometrical picture. The Higgs field is a connection, which arises from the geometry of the two-point set [6,7], see also [4,8–15], and references therein. This suggests that the discrete geometry may play an important role in physics. Differential calculus and gauge theory on discrete groups were proposed by Sitarz in [16], see also [17–21]. The differential calculus on arbitrary finite or countable sets was formulated by Dimakis and Müller-Hoissen in [22,23]. Especially, the generalized $U(1)$ gauge theory above a discrete space with n points ($n > 2$), was briefly discussed by Cammarata and Coquereaux in [24]. For related developments, see [25–28] and references therein. For the metric properties of a finite set, see [29] and references therein.

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In Connes' noncommutative geometry, all the geometrical data are determined by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra, $\mathcal{H}$ is a Hilbert space with an involutive representation π of $\mathcal{A}$, and D is a self-adjoint operator acting on $\mathcal{H}$.

In this paper, we first review the differential calculus on a n-point set [22–24]. We then apply Connes' spectral triple to this formalism to obtain the explicit form of the action functional of $U(1)$ gauge field on the n-point set. Finally, we construct the $U(1)$ gauge theory on a disconnected manifold consisting of n copies of a connected manifold. The explicit action functional in this case is also obtained.

2. Differential calculus on *n*-point set

In this section, we shall review the differential calculus on n-point set. More detailed account of the construction can be found in [22–24].

Let M be a set of n points $i_1, \dots, i_n$ $(n < \infty)$, and $\mathcal{A}$ an involutive algebra of complex functions on M with $(fg)(i) = f(i)g(i)$. Let $p_i \in \mathcal{A}$ defined by

$$p_i(j) = \delta_{ij}. \tag{1}$$

Then it follows that

$$p^* = p, \qquad p_i p_j = \delta_{ij} p_j, \qquad \sum_i p_i = \mathbf{1}, \tag{2}$$

where $\mathbf{1}(i) = 1$. In other words, p_i is a projector in $\mathcal{A}$. Each $f \in \mathcal{A}$ can be written as

$$f = \sum_i f(i) p_i, \tag{3}$$

where $f(i) \in \mathbf{C}$ is a complex number. The algebra $\mathcal{A}$ can be extended to a universal differential algebra $\Omega(\mathcal{A}) = \oplus_{r=0}^{\infty} \Omega^r(\mathcal{A})$ (where $\Omega^0(\mathcal{A}) = \mathcal{A}$) via the action of a linear operator $\mathrm{d} : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$ satisfying

$$\mathrm{d}\mathbf{1} = 0, \qquad \mathrm{d}^2 = 0, \qquad \mathrm{d}(\omega_r \omega') = (\mathrm{d}\omega_r)\omega' + (-1)^r \omega_r \,\mathrm{d}\omega',$$

where $\omega_r \in \Omega^r(\mathcal{A})$. The spaces $\Omega^r(\mathcal{A})$ of r-forms are $\mathcal{A}$-bimodules. $\mathbf{1}$ is taken to be the unit in $\Omega(\mathcal{A})$. From the above properties, the set of functions p_i satisfy the following relations:

$$p_i \,\mathrm{d}p_j = -(\mathrm{d}p_i) p_j + \delta_{ij} \,\mathrm{d}p_i, \tag{4}$$

$$\sum_i \mathrm{d}p_i = 0. \tag{5}$$

This means that the differential calculus over n-point set M associates with it $n-1$ linear independent differentials. There is a natural geometrical representation associated with M. Let the projectors p_i $(i = 1, \dots, n)$ be the orthonormal base vectors in the Euclidean space $\mathbf{R}^n$. Then M forms the vertices of the $(n-1)$-dimensional hypertetrahedron embedded in $\mathbf{R}^n$.

$\Omega(\mathcal{A})$ is made an involutive algebra by

$$(a_0 \,\mathrm{d}a_1 \cdots \mathrm{d}a_n)^* = \mathrm{d}a_n^* \cdots \mathrm{d}a_1^* \, a_0^*, \tag{6}$$

where $a_0, a_1, \dots, a_n \in \mathcal{A}$. Thus we have $\omega^{**} = \omega$ and $(\omega\eta)^* = \eta^*\omega^*$ for $\omega, \eta \in \Omega(\mathcal{A})$, as required. Notice that if $\alpha \in \Omega^1$, then $(\mathrm{d}\alpha)^* = -\mathrm{d}\alpha^*$.

The universal first-order differential calculus Ω^1 is generated by $p_i\,\mathrm{d}p_j (i \neq j)$, i, $j = 1, 2, \dots, n$. Notice that $p_i\,\mathrm{d}p_i$ is the linear combinations of $p_i\,\mathrm{d}p_j (i \neq j)$.

Ω^1 can be defined as the kernel in the algebra $\mathcal{A} \otimes \mathcal{A}$ of the multiplication map. The dimension of Ω^1 is, therefore, $\dim(A \otimes A) - \dim A = n(n-1)$.

Similarly, the compositions of $p_i\,\mathrm{d}p_j (i \neq j)$, $i, j = 1, 2, \dots, n$, generate the higher order universal differential calculus on M. For example, the universal second-order differential calculus Ω^2 is generated by $p_i\,\mathrm{d}p_j\, p_j\,\mathrm{d}p_k (i \neq j, j \neq k)$, $i, j, k = 1, 2, \dots, n$.

Notice that it is much simpler to say that Ω^2 is generated by $p_i\,\mathrm{d}p_j\,\mathrm{d}p_k (i \neq j, j \neq k$ and $i, j, k = 1, 2, \dots, n)$. But the form $p_i\,\mathrm{d}p_j\, p_j\,\mathrm{d}p_k$ is more convenient for us to apply the spectral triple to $\Omega(\mathcal{A})$.

Since $\Omega^p = \Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1$ (p terms), therefore the dimension of Ω^p is $n^p(n-1)^p/n^{p-1} = n(n-1)^p$.

A simple calculation shows that

$$\mathrm{d}p_i = \sum_j (p_j\,\mathrm{d}p_i - p_i\,\mathrm{d}p_j). \tag{7}$$

Furthermore,

$$\mathrm{d}p_i\,\mathrm{d}p_j = \sum_k (p_k\,\mathrm{d}p_i\, p_i\,\mathrm{d}p_j - p_i\,\mathrm{d}p_k\, p_k\,\mathrm{d}p_j + p_i\,\mathrm{d}p_j\, p_j\,\mathrm{d}p_k). \tag{8}$$

Any 1-form α can be written as $\alpha = \sum_{i,j} \alpha_{ij} p_i\,\mathrm{d}p_j$ with $\alpha_{ij} \in \mathbf{C}$ and $\alpha_{ii} = 0$. Especially, $\alpha^* = -\sum_{i,j} \bar{\alpha}_{j,i} p_i\,\mathrm{d}p_j$. Then $\alpha^* = -\alpha$ if and only if $\bar{\alpha}_{ji} = \alpha_{ij}$.

One can find

$$\mathrm{d}\alpha = \sum_{i,j,k} (\alpha_{jk} - \alpha_{ik} + \alpha_{ij}) p_i\,\mathrm{d}p_j\, p_j\,\mathrm{d}p_k. \tag{9}$$

In this paper, we only consider the $U(1)$ gauge field α on M, i.e., α is a connection on M, α is a 1-form, and skew-adjoint, $\alpha^* = -\alpha$, i.e., $\bar{\alpha}_{ji} = \alpha_{ij}$. α obeys the usual transformation rule,

$$\alpha' = u\alpha u^* + u\,\mathrm{d}u^*.$$

Here $u = \sum_i u(i) p_i \in \mathcal{A}$, and $u(i) \in U(1)$, the Abelian unitary group. In order to make the formulae concise, one introduces

$$a = \sum_{i,j} a_{ij} p_i\,\mathrm{d}p_j = \sum_{i,j} (1 + \alpha_{ij}) p_i\,\mathrm{d}p_j \tag{10}$$

with $a_{ii} = 1$. One then has

$$a' = uau^*, \qquad a'_{ij} = u(i) a_{ij} u(j)^*. \tag{11}$$

The curvature of the connection α is given by

$$\theta = \mathrm{d}\alpha + \alpha^2,$$

and transforms in the usual way, $\theta' = u\theta u^*$. Notice that $(\mathrm{d}\alpha)^* = -\mathrm{d}\alpha^* = \mathrm{d}\alpha$ and $(\alpha^2)^* = \alpha^2$, therefore one has $\theta = \theta^*$. As a 2-form, θ can be written as

$$\theta = \sum_{i,j,k} \theta_{ijk} p_i \,\mathrm{d}p_j\, p_j \,\mathrm{d}p_k, \qquad \theta_{ijk} = a_{ij}a_{jk} - a_{ik}. \tag{12}$$

3. From spectral triple to action functional over M

The availability of the spectral triple allows us to project from the algebra of universal forms $\Omega(\mathcal{A})$ to a more useful graded differential algebra.

We now construct the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over the n-point set M [30]. In this case, $\mathcal{A}$ is the algebra on M defined in the last section. Without loss of generality, $\mathcal{H}$ is taken to be a n-dimensional linear space over $\mathbf{C}$, i.e., $\mathcal{H}$ is just the direct sum $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$, $\mathcal{H}_i = \mathbf{C}$. The action of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$\pi(f) = \begin{pmatrix} f(1) & 0 & \dots & 0 \\ 0 & f(2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f(n) \end{pmatrix},$$

where $f \in \mathcal{A}$. Then D is the Hermitian $n \times n$ matrix with elements $D_{ij} = \bar{D}_{ji}$, and D_{ij} is a linear mapping from $\mathcal{H}_j$ to $\mathcal{H}_i$. The following equality defines an involutive representation of $\Omega(\mathcal{A})$ in $\mathcal{H}$,

$$\pi(a_0\,\mathrm{d}a_1 \cdots \mathrm{d}a_n) = i^n \pi(a_0)[D, \pi(a_1)] \cdots [D, \pi(a_n)], \tag{13}$$

where $a_0, a_1, \dots, a_n \in \mathcal{A}$. To ensure the differential d satisfies

$$\mathrm{d}^2 = 0, \tag{14}$$

one has to impose the following condition on D,

$$D^2 = \mu^2 I, \tag{15}$$

where μ is a real constant and I the $n \times n$ unit matrix.

We now take $D_{ij} \neq 0\,(i \neq j)$. Then the representation $\pi : \Omega(\mathcal{A}) \to \mathcal{L}(\mathcal{H})$ is injective on $\Omega(\mathcal{A})$. One can prove that π homomorphism is a differential one, i.e., $\pi(\Omega(\mathcal{A}))$ is well defined.

In Connes' terminology [2], our spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is odd (except the case of $n = 2$).

The projector p_i can be expressed as the $n \times n$ matrix,

$$(\pi(p_i))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i}. \tag{16}$$

Notice that the diagonal elements of D commute exactly with the action of $\mathcal{A}$. For the sake of convenience, we can ignore the diagonal elements of D, i.e.,

$$D_{ii} = 0. \tag{17}$$

From (13) and (16), one has

$$(\pi(p_i\,\mathrm{d}p_j))_{\alpha\beta} = i\delta_{\alpha i}\delta_{\beta j}D_{ij}, \tag{18}$$

$$(\pi(p_i\,\mathrm{d}p_j\;p_j\,\mathrm{d}p_k\;p_k\,\mathrm{d}p_l\;p_l\,\mathrm{d}p_r))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta r}D_{ij}D_{jk}D_{kl}D_{lr}. \tag{19}$$

One can define an inner product $\langle|\rangle$ in $\pi(\Omega(\mathcal{A}))$ by setting

$$\langle\alpha|\beta\rangle = \mathrm{tr}(\alpha^*\beta).$$

Then the action functional of the curvature θ is

$$S = \|\pi(\theta)\|^2 = \langle\pi(\theta)|\pi(\theta)\rangle = \mathrm{tr}(\pi(\theta))^2. \tag{20}$$

From (12), (19) and (20), we have

$$S = \sum_{i,j,k,l} \theta_{ijk}\theta_{kli}D_{ij}D_{jk}D_{kl}D_{li}. \tag{21}$$

Denote

$$a_{ij}D_{ij} = H_{ij}, \tag{22}$$

where a_{ij} is defined in (10). Then $H = (H_{ij})$ is a Hermitian matrix with

$$H_{ii} = 0. \tag{23}$$

From (12), (15), (21), –(23), one thus has

$$S = \mathrm{tr}\,H^4 - 2\mu^2\,\mathrm{tr}\,H^2 + n\mu^4. \tag{24}$$

From (23), the eigenvalues λ_i $(i = 1, 2, \dots, n)$ of H satisfy:

$$\sum_{i=1}^{n} \lambda_i = 0. \tag{25}$$

Eq. (24) can be written as the following:

$$S = \sum_{i=1}^{n} \lambda_i^4 - 2\mu^2 \sum_{i=1}^{n} \lambda_i^2 + n\mu^4. \tag{26}$$

From the quadric expression, (26) can take the form

$$S = C_2^{ijkl} \sum_{i,j,k,l=1}^{n-1} \varphi_i\varphi_j\varphi_k\varphi_l - C_1 \sum_{i=1}^{n-1} \varphi_i^2 + n\mu^4,$$

where $(\varphi_1, \dots, \varphi_{n-1})$ is a vector in $(n-1)$-dimensional Euclidean space $\mathbf{R}^{n-1}$, C_2^{ijkl} and C_1 are real constants.

For the sake of convenience, we identify $\mathbf{R}^{n-1}$ with a subspace embedded in the n-dimensional geometrical representation space of M introduced in Section 2 from now on. In the $(n-1)$-dimensional rectangular coordinate system, the reference point is taken to be the center of the $(n-1)$-dimensional hypertetrahedron. M can then be represented by a set of n vectors in $\mathbf{R}^{n-1}$: e_i^α ($\alpha = 1, \dots, n$; $i = 1, \dots, n-1$), such that

$$\sum_i e_i^\alpha e_i^\beta = \frac{n}{n-1}\delta^{\alpha\beta} - \frac{1}{n-1}. \tag{27}$$

In (27) we have chosen the normalization of the vectors to be unity for convenience. This set of e's satisfy

$$\sum_\alpha e_i^\alpha = 0, \tag{28}$$

$$\sum_\alpha e_i^\alpha e_j^\alpha = \frac{n}{n-1}\delta_{ij}. \tag{29}$$

It should be mentioned that the properties of M are encoded in those of the set of spin states in the Potts model [31]. The reason will be discussed at the end of this section.

Using (28), the eigenvalues of H are

$$\lambda_\alpha = \sum_{i=1}^{n-1} \phi_i e_i^\alpha, \quad \alpha = 1, \dots, n. \tag{30}$$

Here each ϕ_i ($i = 1, \dots, n-1$) is a real parameter. We call $\mathbf{\Phi} = (\phi_1, \dots, \phi_{n-1})$ the order parameter field in $\mathbf{R}^{n-1}$. Finally, from (24), (29) and (30), we obtain the explicit form of S over the n-point set M:

$$S = \sum_{i,j,k,l}\left(\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha\right)\phi_i\phi_j\phi_k\phi_l - \frac{2n}{n-1}\mu^2\left(\sum_i \phi_i^2\right) + n\mu^4. \tag{31}$$

Remark.

1) The gauge field is a $n \times n$ matrix. Meanwhile, the number of the reduced field variables ϕ_i is $n-1$. Notice also that ϕ_i ($i = 1, 2, \dots, n-1$) are real numbers.
2) When $n = 2$, (31) changes into

$$S = 2(\phi^2 - \mu^2)^2. \tag{32}$$

This is just the Hamiltonian density of the Landau phenomenological theory of phase transitions below the critical temperature [32]. Here ϕ is known as the order parameter. Notice that the size of the coefficients of ϕ^2 and ϕ^4 does not affect the values of critical exponents of phase transitions, but it may affect the mass value of the Higgs field when (32) is considered as the Higgs potential: From (15), (22) and (30), we have $\phi^2 = \mu^2|a_{12}|^2$. One then has

$$S = 2\mu^4(|a_{12}|^2 - 1)^2, \tag{33}$$

which is the form of Connes' version of Higgs potential.

3) Eq. (32) is contained in the potential energy density of the continuous-spin formulation of the Ising model (see for example [33,34]). Notice that the Ising model is just the 2-state Potts model. In general, (31) is contained in the potential energy density of the continuous-spin formulation of the n-state Potts model [31]. Since the set of states of the Potts model forms a n-point set, one then can build $U(1)$ gauge theory on this n-point set.

4. Action functional on n copies of a manifold

Let V be an oriented smooth manifold and M, as the previous sections, a n-point set. Then $V \times M$ is a disconnected manifold consisting of n copies of V. Now we construct the $U(1)$ gauge field theory on $V \times M$. Let g be a complex function on $V \times M$. Just as in Section 2, it can be written as

$$g = \sum_i g(i) p_i. \tag{34}$$

Notice that this time $g(i)$ is a complex function over V_i, the ith copy of V.

The algebra that we use will be the tensor product of the De Rham complex of V and of the universal differential algebra of the finite point set M.

Denote the differential on M by $\mathrm{d_f}$. In other words, The differential d in Sections 2 and 3 is replaced by $\mathrm{d_f}$. Let $\mathrm{d_s}$ be the usual differential on V, and d the total differential on $V \times M$. One then has

$$\mathrm{d} = \mathrm{d_s} + \mathrm{d_f}. \tag{35}$$

The nilpotency of d requires that

$$\mathrm{d_s}\,\mathrm{d_f} = -\mathrm{d_f}\,\mathrm{d_s}. \tag{36}$$

Differentiating (41), we have

$$\mathrm{d}g = \sum_i (\mathrm{d_s}\, g(i)) p_i + \sum_i g(i)\, \mathrm{d_f}\, p_i.$$

Any 1-form α can be written as

$$\alpha = \sum_{i,j} \alpha_{ij} p_i\, \mathrm{d_f}\, p_j + \sum_i \alpha_i p_i \tag{37}$$

with α_{ij}, a complex function on V and $\alpha_{ii} = 0$; α_i, a 1-form on V_i.

Now we consider a connection α over $V \times M$, α is a 1-form and skew-adjoint, i.e., α is given by (37) and $\alpha^* = -\alpha$. α obeys the usual transformation rule,

$$\alpha' = u\alpha u^* + u\, \mathrm{d}u^*.$$

Here $u = \sum_i u(i) p_i \in \mathcal{A}$, and $u(i) \in U(1)$, the Abelian unitary group on V_i. α is thus called the $U(1)$ gauge field on $V \times M$. One then finds

$$\mathrm{d}\alpha = \sum_{i,j} (\mathrm{d_s}\, \alpha_{ij}) p_i \, \mathrm{d_f}\, p_j + \sum_{i,j,k} (\alpha_{jk} - \alpha_{ik} + \alpha_{ij}) p_i \, \mathrm{d_f}\, p_j p_j \, \mathrm{d_f}\, p_k + \sum_i (\mathrm{d_s}\, \alpha_i) p_i - \sum_i \alpha_i \, \mathrm{d_f}\, p_i,$$

$$\alpha^2 = \sum_{i,j,k} \alpha_{ij}\alpha_{jk} p_i \, \mathrm{d_f}\, p_j p_j \, \mathrm{d_f}\, p_k + \sum_{i,j} \alpha_{ij}(\alpha_i - \alpha_j) p_i \, \mathrm{d_f}\, p_j.$$

Notice that $\alpha_i^2 = \alpha_i \wedge \alpha_i = 0$.

As in Sections 2 and 3, we introduce

$$a = \sum_{i,j} a_{ij} p_i \, \mathrm{d_f}\, p_j = \sum_{i,j} (1 + \alpha_{ij}) p_i \, \mathrm{d_f}\, p_j.$$

The $U(1)$ gauge transformation rule for a is

$$a' = uau^*,$$

i.e., $a'_{ij} = u(i) a_{ij} u(j)^*$.

α_i obeys the usual $U(1)$ gauge transformation rule,

$$\alpha'_i = \alpha_i + u(i)\, \mathrm{d}u(i)^*.$$

The curvature of the connection α is given by

$$\Theta = \mathrm{d}\alpha + \alpha^2.$$

It can be seen that Θ transforms in the usual way, $\Theta' = u\Theta u^*$. As a 2-form, Θ can be written as

$$\Theta = \sum_i (\mathrm{d_s}\, \alpha_i) p_i + \sum_{i,j} (\mathrm{d_s} + \alpha_i - \alpha_j) a_{ij} p_i \, \mathrm{d_f}\, p_j + \sum_{i,j,k} \theta_{ijk} p_i \, \mathrm{d_f}\, p_j p_j \, \mathrm{d_f}\, p_k,$$

$$\theta_{ijk} = a_{ij} a_{jk} - a_{ik}. \tag{38}$$

We see that Θ has a usual differential degree and a finite-difference degree (α, β) adding up to 2. Let us begin with the term in Θ of bi-degree $(2, 0)$:

$$\Theta^{(2,0)} = \sum_i (\mathrm{d_s}\, \alpha_i) p_i, \tag{39}$$

it is the continuous part of the field strength.

Next, we look at the component $\Theta^{(1,1)}$ of bi-degree $(1, 1)$:

$$\Theta^{(1,1)} = \sum_{i,j} (\mathrm{d_s} + \alpha_i - \alpha_j) a_{ij} p_i \, \mathrm{d_f}\, p_j. \tag{40}$$

$\Theta^{(1,1)}$ corresponds to the interaction between V and M. It also obeys the field strength transformation rule, $\Theta'^{(1,1)} = u\Theta^{(1,1)} u^*$.

Finally, we have the component $\Theta^{(0,2)}$ of degree (0, 2):

$$\Theta^{(0,2)} = \sum_{i,j,k} \theta_{ijk} p_i \, \mathrm{d_f} \, p_j p_j \, \mathrm{d_f} \, p_k. \quad (41)$$

$\Theta^{(0,2)}$ corresponds to the field strength over the finite set M.

Just as in Section 3, we use the formula (13) to deal with the finite-difference degrees, i.e.,

$$\pi(\mathrm{d_f}\, p_i) = i[D, \pi(p_i)].$$

We also introduce the Hermitian matrix H,

$$H_{ij} = a_{ij} D_{ij}.$$

We then obtain the action functional over $V \times M$:

$$S = \int_V \mathcal{L} \, \mathrm{d}\nu.$$

The Lagrangian density is given by the following formulae:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0, \quad (42)$$

$$\mathcal{L}_2 = \|\Theta^{(2,0)}\|^2 = \sum_i |\mathrm{d_s}\, \alpha_i|^2, \quad (43)$$

$$\mathcal{L}_1 = \|\Theta^{(1,1)}\|^2 = \sum_{i,j} [(\mathrm{d_s} + \alpha_i - \alpha_j) H_{ij}][(\mathrm{d_s} + \alpha_j - \alpha_i) H_{ji}], \quad (44)$$

$$\begin{aligned}\mathcal{L}_0 &= \operatorname{tr} H^4 - 2\mu^2 \operatorname{tr} H^2 + n\mu^4 \\ &= \sum_{i,j,k,l} \left(\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha \right) \phi_i \phi_j \phi_k \phi_l - \frac{2n}{n-1} \mu^2 \left(\sum_i \phi_i^2 \right) + n\mu^4. \quad (45)\end{aligned}$$

Remark. The term $\mathcal{L}_2$ is the usual term describing the Lagrangian for a $U(1) \times U(1) \times \cdots \times U(1)$ (n terms) connection. H_{ij} ($i \neq j, i, j = 1, 2, \ldots, n$) in $\mathcal{L}_1$ give a mass to some of the $\alpha_i - \alpha_j$ fields.

Example 1. We first consider the simplest case, i.e.,

$$\alpha_i = A, \quad i = 1, 2, \ldots, n.$$

Here A is a $U(1)$ gauge field on V. The physical meaning of the above assumptions is: there exists unique gauge field, i.e., the Maxwell electromagnetic field over all copies of V.

The Lagrangian density is given by the following formulae:

$$\mathcal{L}_2 = \|\Theta^{(2,0)}\|^2 = n|F|^2 = n|\mathrm{d_s}\, A|^2,$$

$$\mathcal{L}_1 = \|\Theta^{(1,1)}\|^2 = \operatorname{tr}(\mathrm{d_s}\, H)^2 = \frac{n}{n-1} \sum_i (\mathrm{d_s}\, \phi_i)^2,$$

and $\mathcal{L}_0$ is the same as the formula (45).

Notice that $\mathcal{L}_1$ coincides with the kinetic energy density of the continuous-spin formulation of the n-state Potts model [31].

Example 2. We consider the case of $n = 3$. The Lagrangian density is given by the following formulae:

the massless term

$$\mathcal{L}_2 = |\mathrm{d_s}\,\alpha_1|^2 + |\mathrm{d_s}\,\alpha_2|^2 + |\mathrm{d_s}\,\alpha_3|^2,$$

the nontrivial mass term

$$\begin{aligned}\mathcal{L}_1 &= 2[(\mathrm{d_s} + \alpha_1 - \alpha_2)H_{12}][(\mathrm{d_s} + \alpha_2 - \alpha_1)H_{21}]\\ &\quad +2[(\mathrm{d_s} + \alpha_1 - \alpha_3)H_{13}][(\mathrm{d_s} + \alpha_3 - \alpha_1)H_{31}]\\ &\quad +2[(\mathrm{d_s} + \alpha_2 - \alpha_3)H_{23}][(\mathrm{d_s} + \alpha_3 - \alpha_2)H_{32}],\end{aligned}$$

and the Higgs–Landau polynomial

$$\mathcal{L}_0 = \tfrac{9}{8}(\phi_1^2 + \phi_2^2)^2 - 3\mu^2(\phi_1^2 + \phi_2^2) + 3\mu^4.$$

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